



Contact Lorentzian Manifolds of Constant Curvature and CR-Manifolds

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Abstract: In this work, we show that a contact Lorentzian manifolds of dimension $(2n + 1), n \geq 2$, has constant sectional curvature k , then the structure $(\bar{M}^{2n+1}, \bar{\phi}, \xi, \eta, \bar{g})$ is Sasakian and $k = -1 = \bar{g}(\xi, \xi)$, where ξ is characteristic vector field. We present some notions of contact Lorentzian structure with a non-degenerate almost CR manifolds. Moreover, we given some an examples of contact Lorentzian manifolds with condition $\text{trace } h^2 = 0$.

Keywords: Contact Lorentzian structures, Lorentzian metrics, Constant sectional curvature, Non-degenerate CR structures.

1. Introduction

In [Calvaruso, 2011] the authar introduced and studied the contact Lorentzian manifolds $(\bar{M}^{2n+1}, \bar{\phi}, \xi, \eta, \bar{g})$, that is, contact manifolds equipped with a Lorentzian metric associated $\bar{g}$. In this study introduce the tensor $h = \frac{1}{2} \mathcal{L}_\xi \bar{\phi} = \frac{1}{2} N^{(3)}$, where $\mathcal{L}$ is a Lie derivative and $\bar{\phi}$ is tensor field of type $(1, 1)$, which plays an important role in geometry of a contact Lorentzian manifolds. In particular, the K-contact Riemannian manifold of odd-dimension $(2n + 1)$ are characterized by the condition $\rho(\xi, \xi) = 2n$, where ρ Ricci curvature [Blair *et al.*, 2010], since it implies

$\text{tr} h^2 = 0$ and so, $h = 0$ because h is diagonalizable. In [Calvaruso, 2011] (see Section 5), we transacted with contact Lorentzian manifolds of constant sectional curvature κ . the contact Pseudo-metric manifolds of constant sectional curvature and CR geometry of *dimension* > 5 introduced by [Perrone, 2014]. In $(2n + 1)$ – dimensional, showing that the value of constant sectional curvature to determine it depends on the casual character of the Reeb vector field ξ . . In [Calvaruso and Perrone, 2010], we showed that any K -contact Lorentzian manifolds is Sasakian and of constant sectional curvature $k = -1$. The main result of Section 5 in [Calvaruso, 2011] obtained by Olszak [Olszak, 1979] in Riemannian settings, if a contact Riemannian manifold $(\bar{M}^{2n+1}, \eta, \bar{g})$, such that $\dim \bar{M} = 2n + 1, n \geq 2$ of constant sectional curvature k , then, $k = \epsilon = \bar{g}(\xi, \xi)$ and $(\bar{M}^{2n+1}, \eta, \bar{g})$, is Sasakian [Calvaruso, and Perrone, 2010], this result was extended to Pseudo-Riemannian setting in [Calvaruso and Perrone, 2010], and extended also in the Lorentzian case in [Calvaruso, 2011]. This paper is organized as follows

In Section 2 we introduce the basic notations of the contact Lorentzian manifolds and almost CR structures. We present a contact Lorentzian manifolds and non- degenerate CR manifolds are given in Section 3. Finally in Section 4 we proof the main results and we given some examples of contact Lorentzian manifolds, of dimension $(2n + 1), n \geq 2$.

2. Preliminaries

In this section we introduce some basic facts for contact Lorentzian manifolds and almost CR structures. The odd-dimensional $(2n + 1)$ manifold $\bar{M}$ is called a contact manifold if it admits a global differential 1-form η , such that $\eta \wedge d\eta \neq 0$, there exist a Reeb vector field ξ , such that

$$\eta(\xi) = 1 \text{ and } d\eta(\xi, X) = \bar{g}(X, \bar{\phi}\xi) = 0.$$

A compatible Lorentzian metric $\bar{g}$ is said to be an associated if there exist a tensor field $\bar{\phi}$ of type $(1, 1)$, satisfies

$$\eta(X) = -\bar{g}(\xi, X), d\eta(X, Y) = \bar{g}(X, \bar{\phi}Y), \bar{\phi}^2 = -I + \eta \otimes \xi,$$

Then η is a contact form on $\bar{M}$ and $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ is called a contact Lorentzian manifold [G. Calvaruso, 2011]. The associated Lorentzian metric $\bar{g}$ satisfies

$$\bar{g}(\bar{\phi}X, \bar{\phi}Y) = \bar{g}(X, Y) + \eta(X)\eta(Y).$$

Let $\bar{\nabla}$ be the induced Levi-Civita connection and Riemannian curvature tensor $\bar{R}$, given by

$$\bar{R}(X, Y) = \bar{\nabla}_{[X, Y]} - [\bar{\nabla}_X, \bar{\nabla}_Y].$$

The tensor $h = \frac{1}{2}\mathcal{L}_\xi \bar{\phi}$, where $\mathcal{L}$ is a Lie derivative, is self-adjoint and satisfies the following properties.

$$\bar{\nabla}_X \xi = -\epsilon \bar{\phi}X - \bar{\phi}hX, h\bar{\phi} = -\bar{\phi}h, h\xi = \text{tr } h = 0 \quad (2.1)$$

Remark that h is self-adjoint and $h\bar{\phi} = -\bar{\phi}h$, then we obtain $\text{trace } h = \text{trace } h\bar{\phi} = 0$.

A contact Lorentzian manifold (M, η, g) is said to be K-contact if ξ is a killing vector field, equivalently, the tensor $h = 0$. and it is said to be Sasakian if the contact Lorentzian manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ is a normal and satisfies $[\bar{\phi}, \bar{\phi}] + 2d\eta \otimes \xi = 0$, that it is equivalently, $(\bar{\nabla}_X \bar{\phi})Y = \bar{g}(X, Y)\xi + \eta(Y)X$.

Any Sasakian Lorentzian manifold is K-contact, for three-dimensional spaces, a K-contact Lorentzian manifold of dimensional spaces, a K -contact Lorentzian manifold of dimensional $2n + 1$, is Sasakian if satisfies

$$\bar{R}(X, Y)\xi = \eta(X)Y + \eta(Y)X, \quad (2.2)$$

where $\bar{R}$ is Riemannian curvature tensor, the equation (2.2) is equivalently,

$$(\bar{\nabla}_X \bar{\phi})Y = \bar{g}(X, Y)\xi + \eta(Y)X.$$

Refer [Takahashi, 1969] for more information of Lorentzian contact manifold.

3. Contact Lorentzian Manifolds And Non-Degenerate CR Manifolds

In this section we show the following

Lemma 3.1. ([Perrone, 2014]) Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be a contact Pseudo-metric manifold, of dimension $2n + 1, n \geq 2$. If $(\bar{M}, \bar{g})$ is of constant sectional curvature $\mathcal{K}$, then $k = -1 = \bar{g}(\xi, \xi)$ and $h^2 = 0$.

Theorem 3.2. Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be a contact Lorentzian manifold, of $(2n + 1)$ -dimensional, $n > 2$. If $(\bar{M}, \bar{g})$ is of constant sectional curvature $_$, then $k = -1 = \bar{g}(\xi, \xi)$ and $(\bar{M}, \eta, \bar{g})$ is Sasakian.

We note that if $h\bar{\phi} = -\bar{\phi}h$, that it is we have $\text{trace } h = \text{trace } \bar{\phi}h = 0$, then, $h^2 = 0$, implies that $h = 0$ that is, the contact Lorentzian manifold is a K-contact. moreover, because $(\bar{M}, \bar{g})$ has constant sectional curvature -1, and

$\bar{R}(X, Y)\xi = -\bar{g}(X, \xi)Y + \bar{g}(Y, \xi)\xi$ that is, $(\bar{\nabla}_X \bar{\phi})Y = \bar{g}(X, Y)\xi + \eta(Y)X$ satisfied.

On the other hand, we introduce a some notions of CR geometry [Dragomir, 2016].

Let $\bar{M}$ be a real $(2n + 1)$ –dimensional smooth manifold and consider that $(\mathcal{H}(\bar{M}), \bar{J})$ be an almost CR structure on $\bar{M}$. An almost CR structure if integrable is called CR structure, that is satisfied the following condition [Perrone, 2014]

$$[\bar{J}X, Y] + [X, \bar{J}Y] \in \mathcal{H}(\bar{M}) \quad \text{and} \quad \bar{J}([\bar{J}X, Y] + [X, \bar{J}Y]) = [\bar{J}X, \bar{J}Y] - [X, Y], \quad (3.1)$$

$\forall X, Y \in \mathcal{H}(\bar{M})$. An almost structure on $\bar{M}$ is a differentiable 1-form θ on $\bar{M}$ such that $\ker \theta = \mathcal{H}(\bar{M})$. an almost CR structure $(\mathcal{H}(\bar{M}), \bar{J}, \theta)$ is called non-degenerate if $L_\theta(X, Y) = (d\theta)(X, \bar{J}Y)$, for any $X, Y \in \mathcal{H}(\bar{M})$ since, θ is a contact form, the condition in (3.1) is equivalent to

$$\bar{J}([\bar{J}X, Y] + [X, \bar{J}Y]) = [\bar{J}X, \bar{J}Y] - [X, Y], \quad \forall X, Y \in \mathcal{H}(\bar{M}) \quad (3.2)$$

Let $(\bar{M}, \mathcal{H}(\bar{M}), \bar{J}, \theta)$ be a non-degenerate almost CR manifold hence, $\bar{\phi} = \bar{J}$ and $\bar{\phi}(T) = 0$ where T is Reeb vector field of θ . then, $\bar{\phi}^2 = -I + \theta \otimes T$, and the metric $\bar{g}$ given by

$$\bar{g}(X, Y) = d\theta(X, \bar{J}Y), \bar{g}(X, T) = 0, \bar{g}(T, T) = -1 \quad (3.3)$$

For all $X, Y \in \mathcal{H}(\bar{M})$, $\bar{g}$ is Lorentzian metric on $\bar{M}$ [S.Dragomir, M.Hasan and F.AlSolamy, 2016]. The structure $(\bar{\phi}, \xi = T, \eta = \theta, \bar{g})$ is a Lorentzian structure on $\bar{M}$.

A contact Lorentzian structure $(\bar{\phi}, \xi, \eta, \bar{g})$ defines a non-degenerate almost CR structure on $\bar{M}$ given by

$(\mathcal{H}(\bar{M}), \bar{J}, \theta)$, where $\mathcal{H}(\bar{M}) = \ker \eta, \theta = \eta$ and $\bar{J} = \bar{\phi}$. For a contact Pseudo-metric manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$, the generalized Tanaka-Webster connection $\bar{\nabla}$ is given by

$$\bar{\nabla}_X Y = \bar{\nabla}_X Y - \eta(X)\bar{\phi}(Y) - \eta(X)\bar{\nabla}_X \xi + \{(\bar{\nabla}_X \eta)Y\}\xi, \quad (3.4)$$

For all $X, Y \in \Gamma(\bar{M})$ and ∇ is the Levi-Civita connection of $(\bar{M}, \bar{g})$ [Calvaruso et al., 2010]. In particular, from (3.4), we get

$$(\bar{\nabla}_X \bar{\phi})Y = (\bar{\nabla}_X \bar{\phi})Y + \{(\bar{\nabla}_X \eta)\bar{\phi}Y\}\xi + \eta(Y)\bar{\phi}(\bar{\nabla}_X \xi). \quad (3.5)$$

4. Proof of Theorem 3.2

Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be an integrable contact Lorentzian-metric manifold with $\dim \bar{M} = 2n + 1, n \geq 2$, of constant sectional curvature K . By Lemma 3.1 $\bar{M}$ has a constant sectional curvature $K = -1$. Note that the integrable condition in (3.2) is equivalent to the vanishing of $\bar{\nabla} \bar{\phi}$. Then, from (3.5) we have

$$(\bar{\nabla}_X \bar{\phi}Y) = -\{(\bar{\nabla}_X \eta)\bar{\phi}Y\}\xi - \eta(Y)\bar{\phi}(\bar{\nabla}_X \xi), \quad (4.1)$$

Where,

$$\{(\bar{\nabla}_X \eta) \bar{\phi} Y\} = -\bar{g}(X, Y) - \eta(X) \eta(Y) + \bar{g}(hX, Y)$$

and

$$\bar{\phi}(\bar{\nabla}_X \xi) = -X + \eta(X) \xi + hX$$

Then, (4.1) becomes $(\bar{\nabla}_X \bar{\phi})Y = \bar{g}(X - hX, Y) \xi + \eta(Y)(X - hX)$ Consequently, for any $X, Y, Z \in \Gamma(\bar{M})$, we have

$$\begin{aligned} (\bar{\nabla}^2_{X,Y} \bar{\phi})Z &= (\bar{\nabla}_X \bar{\nabla}_Y \bar{\phi})(Y, Z) \\ &= \bar{\nabla}_X \left((\bar{\nabla}_Y \bar{\phi})Z \right) - (\bar{\nabla}_{\bar{\nabla}_X Y} \bar{\phi})Z - (\bar{\nabla}_Y \bar{\phi}) \bar{\nabla}_X Z \\ &= \bar{\nabla}_X (\bar{g}(Y - hY, Z) \xi) - \bar{g}(\bar{\nabla}_X Y - h \bar{\nabla}_X Y, Z) \xi \\ &\quad - \bar{g}(Y - hY, \bar{\nabla}_X Z) \xi - \eta(\bar{\nabla}_X Z)(Y - hY). \end{aligned}$$

From the previous formula for $Z \in \mathcal{H}(\bar{M})$ we have

$$(\bar{\nabla}^2_{X,Y} \bar{\phi})Z = \bar{g}((\bar{\nabla}_X h)Y, Z) \xi - \bar{g}(Y - hY, Z)(\bar{\phi}X + \bar{\phi}hX) + \bar{g}(-\bar{\phi}X + \bar{\phi}hX, Z)(Y - hY).$$

Then, we obtain

$$\begin{aligned} - \left((\bar{\nabla}^2_{X,Y} \bar{\phi})Z + \bar{\nabla}^2_{X,Y} \bar{\phi} \right) Z &= -\bar{g}((\bar{\nabla}_Y h)X - (\bar{\nabla}_X h)Y, Z) \xi + \bar{g}(Y - hY, Z)(-\bar{\phi}X + \bar{\phi}hX) \\ &\quad - \bar{g}(-\bar{\phi}X + \bar{\phi}hX, Z)(Y - hY) - \bar{g}(X - hX)(-\bar{\phi}Y + \bar{\phi}hY) + \bar{g}(-\bar{\phi}Y + \bar{\phi}hY, Z)(X - hX) \quad (4.2). \end{aligned}$$

On the other hand, the curvature tensor satisfies the following identity

$$\bar{R}(X, Y) \bar{\phi} = -(\bar{\nabla}^2_{X,Y} \bar{\phi}) + (\bar{\nabla}^2_{Y,X} \bar{\phi}); \quad (4.3)$$

Where, $(\bar{R}(X, Y) \bar{\phi})Z = \bar{R}(X, Y) \bar{\phi} Z - \bar{\phi} \bar{R}(X, Y) Z$.

Now, for any $X, Y, Z \in \mathcal{H}(\bar{M})$, the curvature tensor is given by

$$\bar{R}(X, Y)Z = -\bar{g}(X, Z)Y + \bar{g}(Y, Z)X;$$

then, from (4.2) and (4.3) we get

$$\bar{g}((\bar{\nabla}_Y h)X - (\bar{\nabla}_X h)Y, Z) = \bar{g}(\bar{R}(X, Y) \bar{\phi} Z, \xi) = 0;$$

and thus

$$\begin{aligned} &\bar{g}(Y, Z) \bar{\phi} hX + \bar{g}(hY, Z) \bar{\phi} X - \bar{g}(hY, Z) \bar{\phi} hX \\ &\quad - \bar{g}(\bar{\phi} X, Z) hY - \bar{g}(\bar{\phi} hX, Z) Y + \bar{g}(\bar{\phi} hX, Z) hY \\ &= \bar{g}(X, Z) \bar{\phi} hY + \bar{g}(hX, Z) \bar{\phi} Y - \bar{g}(hX, Z) \bar{\phi} hY \\ &\quad - \bar{g}(\bar{\phi} Y, Z) hX - \bar{g}(\bar{\phi} hY, Z) X + \bar{g}(\bar{\phi} hY, Z) hX; \quad (4.4) \end{aligned}$$

for any $X, Y, Z \in \mathcal{H}(\bar{M})$:

Now, we define the tensor S of type (1,3) on M by

$$\bar{S}(X, Y, Z) = \bar{g}(Y, Z) \bar{\phi} hX + \bar{g}(hY, Z) \bar{\phi} X - \bar{g}(hY, Z) \bar{\phi} hX$$

$$\begin{aligned}
& -\bar{g}(\bar{\phi}X, Z)hY - \bar{g}(\bar{\phi}hX, Z)Y + \bar{g}(\bar{\phi}hX, Z)hY \\
& = \bar{g}(X, Z)\bar{\phi}hY + \bar{g}(hX, Z)\bar{\phi}Y - \bar{g}(hX, Z)\bar{\phi}hY \\
& - \bar{g}(\bar{\phi}Y, Z)hX - \bar{g}(\bar{\phi}hY, Z)X + \bar{g}(\bar{\phi}hY, Z)hX. \quad (4.5)
\end{aligned}$$

From (4.4) we get $\bar{S}(X, Y, Z) = 0$ for any $X, Y, Z \in \mathcal{H}(\bar{M})$.

The tensor $\bar{S}_X(Y, Z) = \bar{S}(X, Y, Z)$ satisfies $\text{trace}_{\bar{g}}\bar{S}_X = 0$. Then, from (4.5) we obtain

$$\begin{aligned}
\text{trace}_{\bar{g}}\bar{S}_X &= 2n\bar{\phi}hX + (\text{trace}_{\bar{g}}h)\bar{\phi}X - (\text{trace}_{\bar{g}}h)\bar{\phi}hX \\
& - h\bar{\phi}X - \bar{\phi}hX + h\bar{\phi}hX - \bar{\phi}hX - \bar{\phi}hX + \bar{\phi}h^2X
\end{aligned}$$

$$+ (\text{trace}_{\bar{g}}\bar{\phi})hX + (\text{trace}_{\bar{g}}(\bar{\phi}h))X - (\text{trace}_{\bar{g}}(\bar{\phi}h))hX.$$

Since $h\bar{\phi} = -\bar{\phi}h$, and $\text{trace}_{\bar{g}}h = \text{trace}_{\bar{g}}(\bar{\phi}h) = \text{trace}_{\bar{g}}\bar{\phi} = 0$.

Furthermore, by Lemma 3.1, $h^2 = 0$. Therefore, $hX = 0$ for any $X \in \mathcal{H}(\bar{M})$, that is (2.2) is satisfied, which complete the proof.

Now, we introduce some examples of contact Lorentzian manifolds. The Paper [Calvaruso, 2011] do not contain an example of contact Lorentzian manifold, in this section, we give an examples in dimension five of contact Lorentzian manifolds.

Suppose that the space $\bar{M} = \mathbb{R}^5(x_1, x_2, x_3, x_4, z)$ and there exist two smooth functions $A, B \in C^\infty(\mathbb{R}^5)$. Now, we can put $\partial_i = \frac{\partial}{\partial x_i}, i = 1, 2, 3, 4$, and $\partial_z = \frac{\partial}{\partial z}$.

The vector fields $\{V_1, V_2, V_3, V_4, V_5 = \xi\}$ on $\mathbb{R}^5$ defined by

$$\begin{aligned}
\sqrt{2}V_1 &= A\partial x_1 + \partial x_2 + \partial x_3 - 2x_1\partial_z, \sqrt{2}V_2 = \partial x_1 - B\partial x_2 - \partial x_4 + 2x_2\partial_z \\
\sqrt{2}V_3 &= -A\partial x_1 + \partial x_2 - \partial x_3 - 2x_1\partial_z, \sqrt{2}V_4 = \partial x_1 + B\partial x_2 + \partial x_4 - 2x_2\partial_z,
\end{aligned}$$

$\xi = \partial_z = V_5$. We define a Lorentzian metric g of signature $(- - + + +)$ by

$$\bar{g}(V_1, V_1) = \bar{g}(V_2, V_2) = -1; \bar{g}(V_3, V_3) = \bar{g}(V_4, V_4) = 1; \bar{g}(V_i, V_j) = 0, \text{ if } i \neq j,$$

$\bar{g}(\xi, \xi) = -1$. The 1-form η is defined by $\eta(X) = -\bar{g}(\xi, X)$, hence,

$$\eta(\partial x_1) = \eta(\partial x_2) = 0, \eta(\partial x_3) = 2x_1, \eta(\partial x_4) = 2x_2, \eta(\xi) = 1. \text{ The tensor}$$

field $\bar{\phi}$ of type $(1, 1)$ define by $\bar{\phi}(V_1) = -V_2, \bar{\phi}(V_2) = V_1, \bar{\phi}(V_3) = -V_4,$

$\bar{\phi}(V_4) = V_3, \bar{\phi}(\xi) = 0$, similarly, $\bar{\phi}(E_1) = E_2, \bar{\phi}(E_2) = -E_1, \bar{\phi}(E_3) = E_4,$

$$\bar{\phi}(E_4) = -E_3, \bar{\phi}(\xi) = 0, \quad \text{where } E_1 = \frac{1}{\sqrt{2}}(V_2 + V_4), E_2 = E_2 = \frac{1}{\sqrt{2}}(V_1 +$$

$$V_3), E_3 = \frac{1}{\sqrt{2}}(V_1 - V_3), E_4 = \frac{1}{\sqrt{2}}(V_4 - V_2), \text{ and } E_i (i = 1, \dots, 4) \text{ are a null vector}$$

fields. The vector fields $\{E_1, E_2, E_3, E_4, \xi\}$, defines a frame on $\mathbb{R}^5$ of $\eta(E_i) = 0, i = 1, \dots, 4$ with respect to frame $\{E_1, E_2, E_3, E_4, E_5, \xi\}$, the 2-form $d\eta$ expressed in the form

$$d\eta = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and the metric $\bar{g}$ given by

$$G = \bar{g}(E_i, E_j) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

Since,

$$\begin{aligned} G\bar{\phi} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = d\eta \end{aligned}$$

That is, $G\bar{\phi} = d\eta$. Then, we obtain that η is a contact form and $(\xi, \bar{\phi}, \eta, \bar{g})$ is a contact Lorentzian structure, furthermore, the tensor $h = \frac{1}{2}E_\xi \bar{\phi}$ satisfies

$$2hE_1 = [\xi, \bar{\phi}E_1] - \bar{\phi}[\xi, E_1] = [\partial_z, \partial x_2] - \bar{\phi}[\partial_z, \partial x_1] = 0,$$

$$2hE_2 = [\xi, \bar{\phi}E_2] - \bar{\phi}[\xi, E_2] = [\partial_z, \partial x_1] - \bar{\phi}[\partial_z, \partial x_2] = 0,$$

$$2hE_3 = [\xi, \bar{\phi}E_3] - \bar{\phi}[\xi, E_3] = [\partial_z, B\partial x_2 + \partial x_4 - 2x_2\partial_z],$$

$$-\bar{\phi}[\partial_z, A\partial x_1 + \partial x_3 - 2\partial x_1\partial_z] = (B - A)_z E_1:$$

Furthermore, $trh^2 = 0$. Moreover, $h = 0$, then, the structure is a K-contact, if

and only if $(B - A) = 0$. From [Perrone, 2014] if $A \neq B$, we obtain a contact Pseudo-
etric

structure with $trh^2 = 0$ and ξ is not killing vector field.

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